



MINISTRY OF EDUCATION, SINGAPORE
in collaboration with
CAMBRIDGE ASSESSMENT INTERNATIONAL EDUCATION
General Certificate of Education Advanced Level
Higher 2

MATHEMATICS

9758/01

Paper 1

For examination from 2025

SPECIMEN PAPER

3 hours

Additional Materials: List of Formulae and Results (MF27)

Suggested solutions.

READ THESE INSTRUCTIONS FIRST

Answer **all** questions.

Write your answers on the Printed Answer Booklet. Follow the instructions on the front cover of the answer booklet.

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.

You are expected to use an approved graphing calculator.

Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.

Where unsupported answers from a graphing calculator are **not** allowed in a question, you must present the mathematical steps using mathematical notations and not calculator commands.

You must show all necessary working clearly.

The number of marks is given in brackets [] at the end of each question or part question.



Singapore Examinations and Assessment Board



Cambridge Assessment
International Education

1

A circular ink-blot is expanding such that the rate of change of its diameter D with respect to time t is 0.3 cm s^{-1} . Find the rate of change of both the circumference and the area of the circle with respect to t when the radius of the circle is 2 cm. [4]

Let C and A be the circumference and area of the circle respectively.

$$C = \pi D \quad \text{and} \quad A = \pi \left(\frac{D}{2}\right)^2 = \frac{\pi}{4} D^2$$

$$\frac{dC}{dD} = \pi \quad \frac{dA}{dD} = \frac{\pi}{2} D$$

Given $\frac{dD}{dt} = 0.3 \text{ cm/s}$, when $r = 2 \Rightarrow D = 4$,

$$\frac{dC}{dt} = \frac{dC}{dD} \times \frac{dD}{dt}$$

$$= \pi \times 0.3$$

$$= 0.3\pi \text{ cm/s} //$$

and

$$\frac{dA}{dt} = \frac{dA}{dD} \times \frac{dD}{dt}$$

$$= \frac{\pi}{2}(4) \times 0.3$$

$$= 0.6\pi \text{ cm}^2/\text{s} //$$

2(a)

The curve C with equation $y = x^3$ is transformed onto the curve with equation $y = f(x)$ by a translation of 2 units in the negative x -direction, followed by a stretch of factor $\frac{1}{4}$ parallel to the y -axis.

Write down the equation of the new curve.

[1]

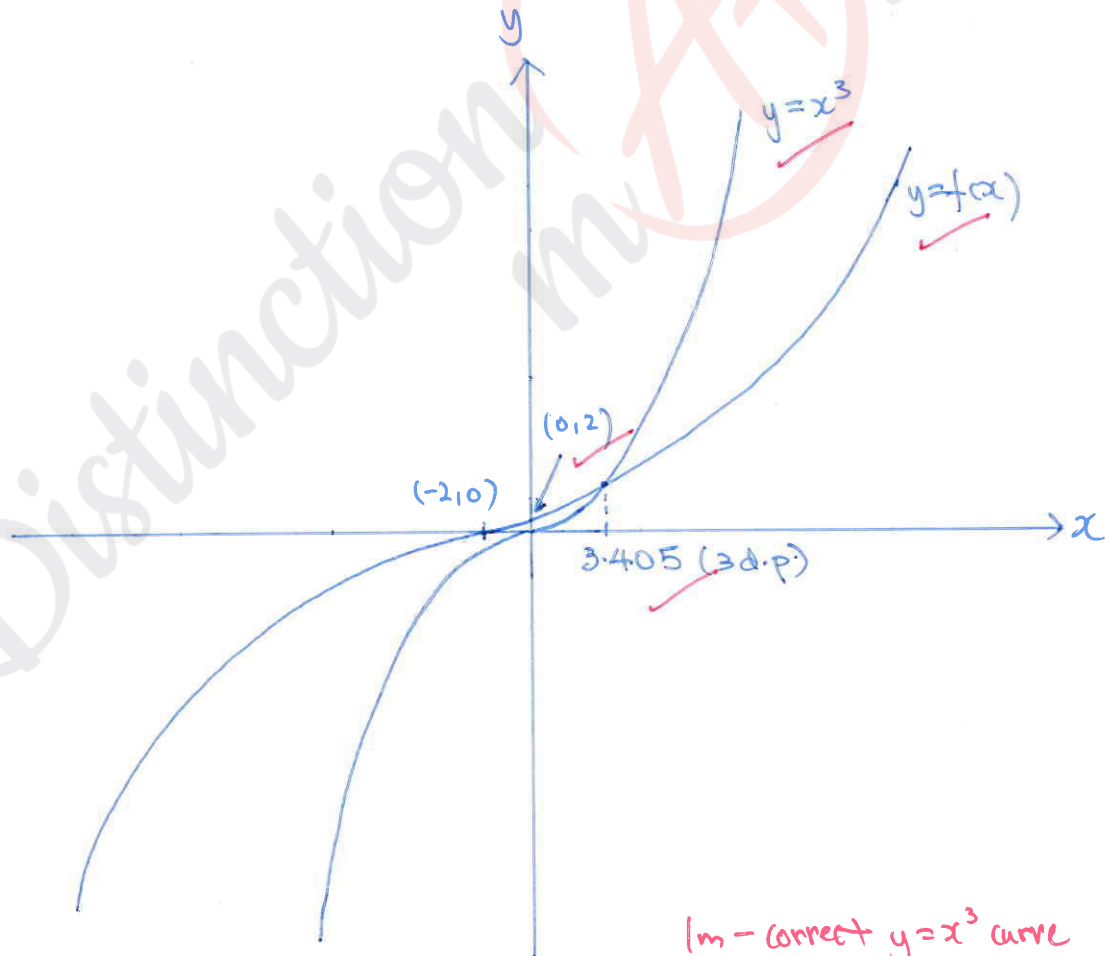
$$y = x^3 \longrightarrow y = (x+2)^3 \longrightarrow 4y = (x+2)^3$$

$\therefore$ Equation of new curve is ~~$y = \frac{1}{4}(x+2)^3$~~

Accept $4y = (x+2)^3$

2(b)

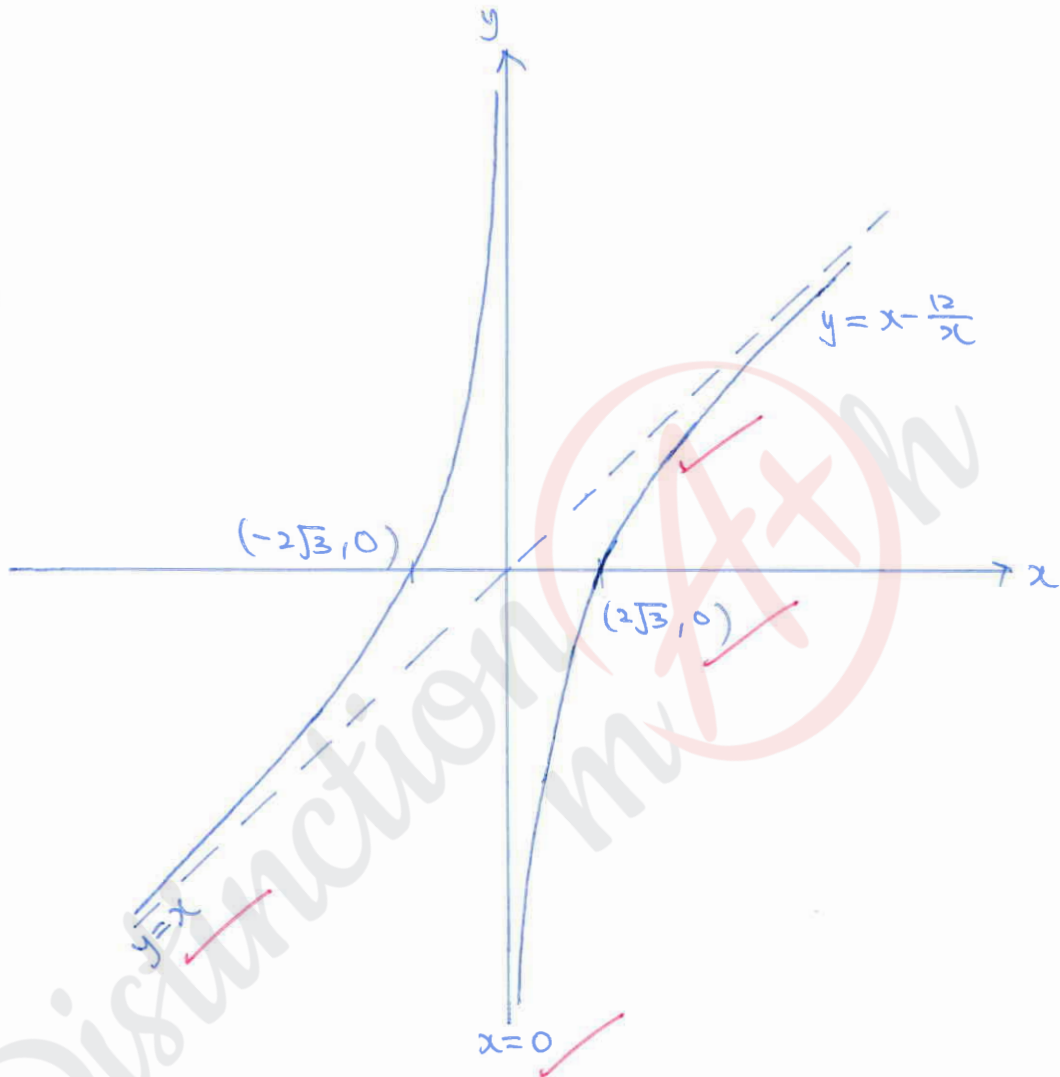
Sketch C and the curve with equation $y = f(x)$ on the same diagram, stating the values of the coordinates of the points where $y = f(x)$ crosses the x - and y -axes. Find the x -coordinate(s) of the point(s) where the two curves intersect, giving your answer(s) correct to 3 decimal places. [4]



1m - correct $y = x^3$ curve
 1m - correct $y = f(x)$ curve
 1m - Both x - and y -coordinates of $y = f(x)$ present.
 1m - x -coordinate of intersection in 3dp.

3(a)

Sketch the curve with equation $y = x - \frac{12}{x}$, giving the exact coordinates of the point(s) where the curve crosses the axes and the equations of any asymptotes. [4]



$$\text{let } y=0, \quad x - \frac{12}{x} = 0$$

$$x^2 = 12$$

$$x = \pm 2\sqrt{3}$$

1m - $y=x$ asymptote

1m - $x=0$ asymptote

1m - Two x -coordinates

1m - correct shape and behavior.

3(b)

Hence, or otherwise, solve the inequality $x - \frac{12}{x} < 1$.

[3]

* {

Insert $y=1$ into the graph.Using G.C., the intersection points of $y=1$ and $y=x - \frac{12}{x}$ are $(-3, 1)$ and $(4, 1)$ ✓ (explained how answers are obtained)For $x - \frac{12}{x} < 1$, $x < -3$ or $0 < x < 4$ //

* Alternatively,

Let $y=1$ to find the intersection points,

$$x - \frac{12}{x} = 1$$

$$x^2 - x - 12 = 0$$

$$(x-4)(x+3) = 0$$

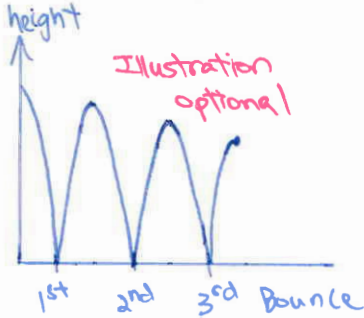
$$\therefore x=4 \text{ or } x=-3$$

4(a)

A student drops a ball vertically onto a hard surface and measures the height reached by the ball after each successive bounce. She drops the ball from an initial height of 160 cm and she estimates that the height the ball reaches after each bounce is $\frac{7}{8}$ of the height reached by the previous bounce.

Find the total distance that the ball has travelled when it reaches the highest point after the third bounce.

[2]



Total distance

$$= 160 + 2\left(\frac{7}{8}\right)(160) + 2\left(\frac{7}{8}\right)^2(160) + \left(\frac{7}{8}\right)^3(160)$$

$$= 792.1875 \text{ m (exact)}$$

4(b)

The ball is considered to have stopped bouncing when a bounce first results in the height the ball reaches being less than 0.01 cm. Find how many bounces the ball has made and the total distance that the ball has travelled in this case. [6]

n^{th} bounce	0	1	2	...	N
max height after the n^{th} bounce	160	$\frac{7}{8}(160)$	$(\frac{7}{8})^2(160)$	...	$(\frac{7}{8})^N(160)$

Let N be the no. of bounces, then

$$\text{max height after the } N^{\text{th}} \text{ bounce} = (\frac{7}{8})^N(160)$$

Able to express height after N^{th} bounce

$$\text{Let } (\frac{7}{8})^N(160) < 0.01$$

$$(\frac{7}{8})^N < \frac{1}{16000}$$

$$N(\ln \frac{7}{8}) < \ln \frac{1}{16000}$$

$$N > (\ln \frac{1}{16000}) \div (\ln \frac{7}{8})$$

$$N > 72.495 \text{ (5sf)}$$

This is a negative value.

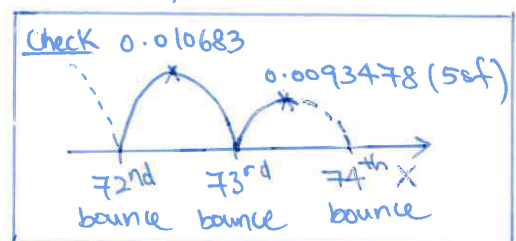
Note: Solving using G.C. can be accepted.

Solve the inequality

$\Rightarrow$ The ball first reaches a max height of less than 0.01 cm after the 73rd bounce.

Hence, the ball bounced 73 times //

Correct no. of bounces.



Total Distance

$$= 160 + 2 \left[\frac{7}{8}(160) + (\frac{7}{8})^2(160) + \dots + (\frac{7}{8})^{72}(160) \right] + (\frac{7}{8})^{73}(160)$$

$$= 160 + 2 \left\{ \frac{\frac{7}{8}(160) [1 - (\frac{7}{8})^{72}]}{1 - \frac{7}{8}} \right\} + (\frac{7}{8})^{73}(160)$$

G.P. sum of first 72 terms.

$$= 2400 \text{ cm (3 s.f.)} //$$

5(a)

The curve C has equation $y = \frac{1}{x}(\ln x)^3$, where $x > 1$.

Find the exact x -coordinate, $x = x_1$, of the turning point on C and determine its nature.

[4]

$$y = \frac{1}{x}(\ln x)^3$$

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{x}(3)(\ln x)^2\left(\frac{1}{x}\right) - \frac{1}{x^2}(\ln x)^3 \\ &= \frac{3}{x^2}(\ln x)^2 - \frac{1}{x^2}(\ln x)^3\end{aligned}$$

$$\text{Let } \frac{dy}{dx} = 0$$

$$\frac{3}{x^2}(\ln x)^2 - \frac{1}{x^2}(\ln x)^3 = 0$$

$$3(\ln x)^2 - (\ln x)^3 = 0$$

$$3 - \ln x = 0 \quad (\text{since } x > 1 \Rightarrow \ln x > 0)$$

$$\therefore \ln x = 3$$

$$x = e^3$$

Using the First Derivative Test,

x	20	e^3	20.1
$\frac{dy}{dx}$	9.58×10^{-3} > 0	0	-1.60×10^{-5} < 0
slope	/	—	\

Hence, at $x = e^3$, the turning point is a maximum point. conclusion

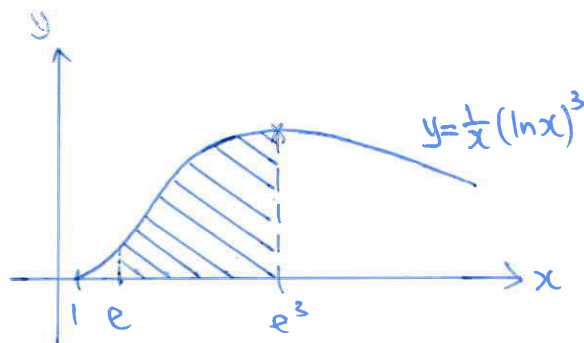
Note: For Second Derivative Test,

$$\frac{d^2y}{dx^2} = \frac{6}{x^3} \ln x - \frac{9}{x^3}(\ln x)^2 + \frac{2}{x^3}(\ln x)^3$$

$$\frac{d^2y}{dx^2} \bigg|_{x=e^3} = -0.00111 < 0 \quad (\text{3sf})$$

5(b)

Using calculus, find the exact area of the region between C , the line $y = 0$ and the lines with equations $x = e$ and $x = e^3$. [3]



$$\text{Area} = \int_e^{e^3} \frac{1}{x} (\ln x)^3 dx$$

$$= \left[\frac{1}{4} (\ln x)^4 \right]_e^{e^3}$$

$$= \frac{1}{4} (\ln e^3)^4 - \frac{1}{4} (\ln e)^4$$

$$= \frac{1}{4} (3)^4 - \frac{1}{4} (1)^4$$

$$= 20 \text{ units}^2 //$$

use:

$$\int f'(x) [f(x)]^n dx$$

$$= \frac{[f(x)]^{n+1}}{n+1} + C.$$

6(a)

The non-zero vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are such that $\mathbf{a} \times \mathbf{b} = \mathbf{c} \times \mathbf{a}$. Given that $\mathbf{b} \neq -\mathbf{c}$, find a linear relationship between $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$. [3]

$$\underline{\underline{\mathbf{a}}} \times \underline{\underline{\mathbf{b}}} = \underline{\underline{\mathbf{c}}} \times \underline{\underline{\mathbf{a}}}$$

$$\underline{\underline{\mathbf{b}}} \neq -\underline{\underline{\mathbf{c}}} \\ \Rightarrow \underline{\underline{\mathbf{b}}} + \underline{\underline{\mathbf{c}}} \neq \underline{\underline{\mathbf{0}}}$$

$$\underline{\underline{\mathbf{a}}} \times \underline{\underline{\mathbf{b}}} - \underline{\underline{\mathbf{c}}} \times \underline{\underline{\mathbf{a}}} = \underline{\underline{\mathbf{0}}}$$

$$\underline{\underline{\mathbf{a}}} \times \underline{\underline{\mathbf{b}}} + \underline{\underline{\mathbf{a}}} \times \underline{\underline{\mathbf{c}}} = \underline{\underline{\mathbf{0}}}$$

$$\underline{\underline{\mathbf{a}}} \times (\underline{\underline{\mathbf{b}}} + \underline{\underline{\mathbf{c}}}) = \underline{\underline{\mathbf{0}}} \quad \checkmark$$

Since $\underline{\underline{\mathbf{a}}} \neq \underline{\underline{\mathbf{0}}}$ and $\underline{\underline{\mathbf{b}}} + \underline{\underline{\mathbf{c}}} \neq \underline{\underline{\mathbf{0}}}$, ← Need to establish this statement

$$\therefore \underline{\underline{\mathbf{a}}} \parallel (\underline{\underline{\mathbf{b}}} + \underline{\underline{\mathbf{c}}}) \quad \checkmark$$

$$\text{Hence } \underline{\underline{\mathbf{a}}} = \lambda(\underline{\underline{\mathbf{b}}} + \underline{\underline{\mathbf{c}}}), \text{ where } \lambda \in \mathbb{R} \setminus \{0\}. \quad \checkmark$$

6(b)(i)

The variable position vector $\mathbf{v}$ satisfies the equation $\mathbf{v} \times (\mathbf{i} - 3\mathbf{k}) = 2\mathbf{j}$.Find the set of position vectors $\mathbf{v}$.

[3]

$$\underline{v} \times (\underline{i} - 3\underline{k}) = 2\underline{j}$$

Let $\underline{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ where x, y, z are real.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -3y \\ 3x + z \\ -y \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$$

$$\therefore -3y = 0 \Rightarrow y = 0$$

$$3x + z = 2$$

$$-y = 0 \Rightarrow y = 0$$

$$\text{Let } z = \mu, \mu \in \mathbb{R}$$

$$\Rightarrow x = \frac{1}{3}(2 - \mu)$$

$$\therefore \underline{v} = \frac{1}{3}(2 - \mu)\underline{i} + \mu\underline{k}$$

$$\text{Hence, } \left\{ \mu \in \mathbb{R} : \frac{1}{3}(2 - \mu)\underline{i} + \mu\underline{k} \right\}$$

Alternatively,

Let $x = \mu, \mu \in \mathbb{R}$

$$\Rightarrow z = 2 - 3\mu$$

$$\underline{v} = \mu\underline{i} + (2 - 3\mu)\underline{k}$$

$$\left\{ \mu \in \mathbb{R} : \mu\underline{i} + (2 - 3\mu)\underline{k} \right\}$$

or equivalent.

6(b)(ii)

Describe the set of points represented by position vectors $\mathbf{v}$ geometrically.

[2]

$$\underline{v} = \frac{1}{3}(2 - \mu)\underline{i} + \mu\underline{k}, \mu \in \mathbb{R}$$

$$= \frac{2}{3}\underline{i} - \frac{1}{3}\mu\underline{i} + \mu\underline{k}$$

$$= \frac{2}{3}\underline{i} + \mu\left(-\frac{1}{3}\underline{i} + \underline{k}\right)$$

$$= \begin{pmatrix} \frac{2}{3} \\ 0 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} -\frac{1}{3} \\ 0 \\ 1 \end{pmatrix}$$

Hence the set of points form a straight line passing through $\left(\frac{2}{3}, 0, 0\right)$ with direction vector $\begin{pmatrix} -\frac{1}{3} \\ 0 \\ 1 \end{pmatrix}$.

conclusion.

Alternatively,

$$\underline{v} = 2\underline{k} + \mu(\underline{i} - 3\underline{k})$$

$$= \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix}$$

$\Rightarrow$ set of points form a straight line passing through $(0, 0, 2)$ with direction vector $\begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix}$.

7(a)

Do not use a calculator in answering this question.

Showing your working, find the complex numbers z and w which satisfy the simultaneous equations.

$$\begin{aligned} 2iz + (1 - 2i)w &= 4 \\ (1 + i)z + (2 + i)w &= 3 \end{aligned}$$

[6]

$$2iz + (1 - 2i)w = 4$$

$$2i^2z + (i - 2i^2)w = 4i \quad \swarrow \times i$$

$$-2z + (2 + i)w = 4i$$

$$(2 + i)w = 2z + 4i \quad \text{--- (1)}$$

$$(1 + i)z + (2 + i)w = 3 \quad \text{--- (2)}$$

sub (1) into (2)

$$(1 + i)z + 2z + 4i = 3$$

$$(3 + i)z = 3 - 4i$$

$$\begin{aligned} z &= \frac{3 - 4i}{3 + i} \times \frac{3 - i}{3 - i} \\ &= \frac{9 - 3i - 12i + 4i^2}{3^2 - i^2} \\ &= \frac{9 - 4 - 15i}{9 + 1} \end{aligned}$$

$$= \frac{5 - 15i}{10} = \frac{1 - 3i}{2}$$

$$\text{sub } z = \frac{1 - 3i}{2} \text{ into (1):}$$

$$(2 + i)w = 2 \left(\frac{1 - 3i}{2} \right) + 4i$$

$$(2 + i)w = 1 + i$$

$$\begin{aligned} w &= \frac{1 + i}{2 + i} \times \frac{2 - i}{2 - i} \\ &= \frac{2 - i + 2i - i^2}{2^2 - i^2} \end{aligned}$$

$$= \frac{2 + 1 + i}{4 + 1}$$

$$= \frac{3 + i}{5}$$

$$\text{or } \frac{3}{5} + \frac{1}{5}i$$

setting up one eqⁿ correctly for substitution (or equivalent)

apply substitution correctly

rationalising the denominator

$$\text{or: } \frac{1}{2} - \frac{3}{2}i$$

rationalising the denominator

7(b)(i)
and
7(b)(ii)

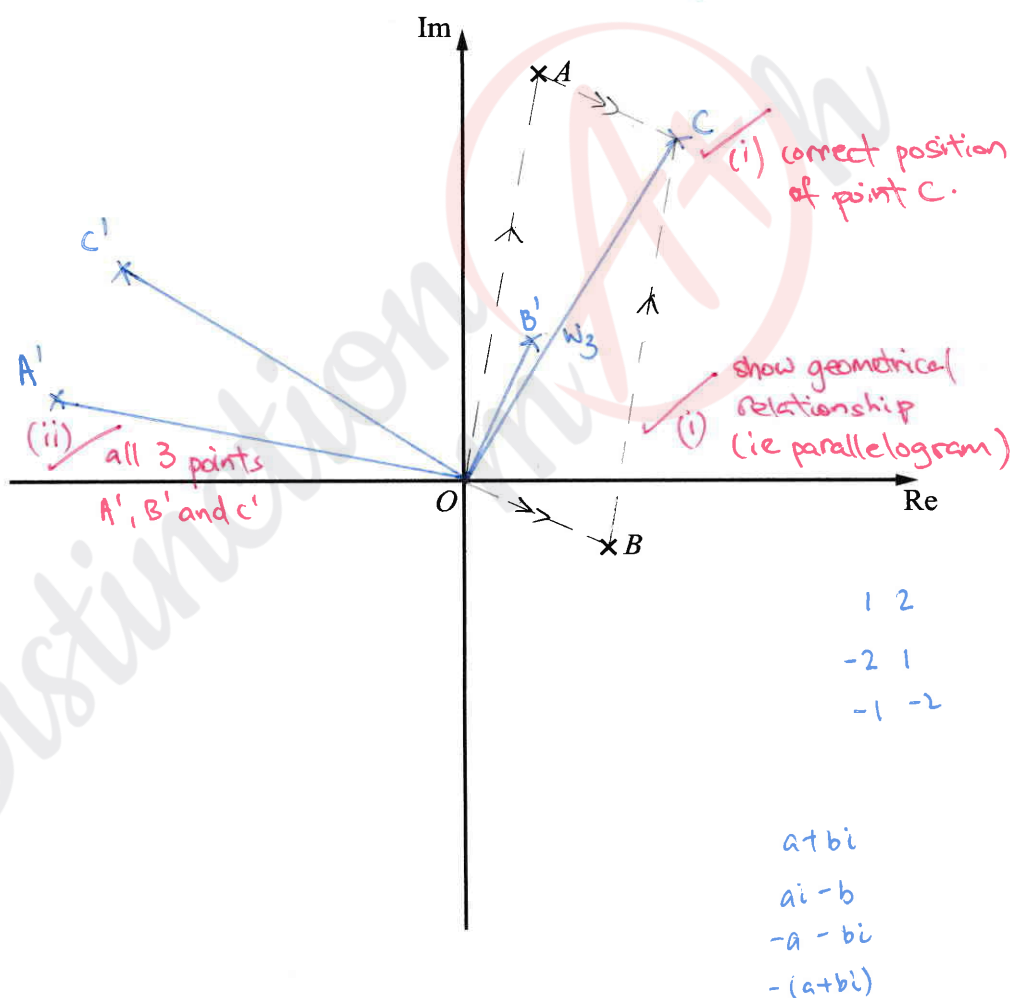
The points A and B on the Argand diagram represent the complex numbers w_1 and w_2 respectively.

- (i) On the copy of the Argand diagram in the Printed Answer Booklet, plot the point C to represent the complex number w_3 , where $w_3 = w_1 + w_2$. Show clearly the geometrical relationship between the points A , B and C . [2]

The points A' , B' and C' represent the complex numbers iw_1 , iw_2 and iw_3 respectively.

- (ii) On the same Argand diagram in the Printed Answer Booklet, plot the points A' , B' and C' . State the transformation that maps the points A , B and C onto the points A' , B' and C' . [2]

ii) Rotate points A, B and C by 90° anti-clockwise about the Origin to map onto points A', B' and C' . (ii)



7(b)(iii)

The transformation in part (ii) maps point A' onto the point A'' . Determine, with justification, whether A'' represents the complex conjugate of w_1 . [1]

No, A'' does not represent w_1^* . It is in fact $-w_1$.

Let $w_1 = a+bi$, $a, b \in \mathbb{R} \Rightarrow w_1^* = a-bi$

A' : $iw_1 = i(a+bi)$

A'' : $i^2 w_1 = i^2(a+bi) = -(a+bi) = -w_1 \neq w_1^*$

Must include justification - (Plotting A'' on diagram above is also accepted)

8(a)

The curve C has parametric equations

$$x = \cos^3 t, \quad y = \sin^3 t, \quad \text{where } 0 < t < \frac{1}{2}\pi.$$

Find the equation of the tangent to C at the point P with parameter p .

[3]

$$x = \cos^3 t \Rightarrow \frac{dx}{dt} = -3\cos^2 t \sin t$$

$$y = \sin^3 t \Rightarrow \frac{dy}{dt} = 3\sin^2 t \cos t$$

$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{3\sin^2 t \cos t}{-3\cos^2 t \sin t} = -\frac{\sin t}{\cos t} = -\tan t$$

$$\text{When } t = p, \quad x = \cos^3 p, \quad y = \sin^3 p \quad \text{and} \quad \frac{dy}{dx} = -\tan p$$

Equation of tangent to C at point P :

$$y - \sin^3 p = -\tan p (x - \cos^3 p)$$

$$y = (-\tan p)x + \frac{\sin p}{\cos p} \cos^3 p + \sin^3 p$$

$$y = (-\tan p)x + \sin p \cos^2 p + \sin p (1 - \cos^2 p)$$

$$y = (-\tan p)x + \sin p \quad // \quad \text{simplified to this}$$

8(b)

The tangent at P meets the x -axis at the point A and meets the y -axis at the point B . Find the length of AB .

[3]

$$y = (-\tan p)x + \sin p$$

$$\text{When } x = 0, \quad y = \sin p \quad \therefore B(0, \sin p)$$

$$\text{When } y = 0, \quad (\tan p)x = \sin p$$

$$x = \sin p \div \tan p$$

$$= \sin p \div \frac{\sin p}{\cos p}$$

$$= \cos p. \quad \therefore A(\cos p, 0)$$

obtained
either point
A or B gives
1 mark.

$$\text{length } AB = \sqrt{(0 - \cos p)^2 + (\sin p - 0)^2}$$

$$= \sqrt{\cos^2 p + \sin^2 p} \quad \text{reduced to } \sqrt{\cos^2 p + \sin^2 p}$$

$$= \sqrt{1}$$

$$= 1 \text{ unit.}$$

8(c)

Show that the cartesian equation of C may be written as $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1$.

[2]

$$\left[\begin{array}{l} x = \cos^3 t \\ \cos^2 t = (\sqrt[3]{x})^2 = x^{\frac{2}{3}} \end{array} \right. \quad \left. \begin{array}{l} y = \sin^3 t \\ \sin^2 t = (\sqrt[3]{y})^2 = y^{\frac{2}{3}} \end{array} \right]$$

Both conversion

$$\left[\begin{array}{l} \text{Since } \cos^2 t + \sin^2 t = 1 \\ x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1 \text{ (shown)} \end{array} \right]$$

must include identity statement

8(d)

The region bounded by C , the line $x = 0$ and the line $y = 0$ is rotated through 2π about the y -axis. Find the volume of revolution of the solid formed, giving your answer correct to 3 decimal places.

[4]

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1$$

$$x^{\frac{2}{3}} = 1 - y^{\frac{2}{3}}$$

$$x^2 = (1 - y^{\frac{2}{3}})^3$$

correct integrand setup.

Volume generated

$$= \pi \int_0^1 (1 - y^{\frac{2}{3}})^3 dy$$

correct upper and lower bound.

wrote π down

$$= 0.479 \text{ units}^3 \text{ (3dp)}$$

(from G.C.)

since $0 < t < \frac{\pi}{2}$

$$0 < \cos t < 1$$

$$0 < \cos^3 t < 1$$

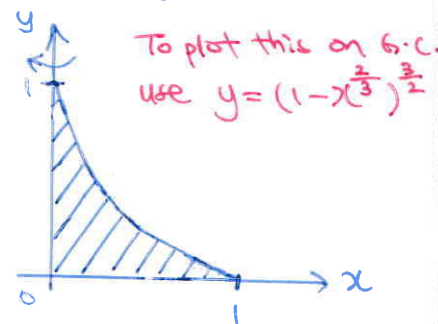
$$\Rightarrow 0 < x < 1$$

since $0 < t < \frac{\pi}{2}$

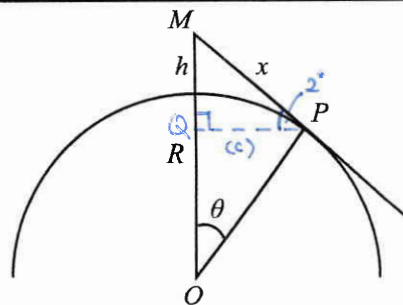
$$0 < \sin t < 1$$

$$0 < \sin^3 t < 1$$

$$\Rightarrow 0 < y < 1$$



9(a)



A man M is at the top of a mountain which is of height h km. The radius of the earth is assumed to be a constant R km. The furthest point on the earth's surface that the man can see is a point P such that $MP = x$ km and the angle $POM = \theta$, where O is the centre of the earth (see diagram). You may assume that the height of the man is negligible.

Show that $x = (2hR)^{\frac{1}{2}} \left(1 + \frac{h}{2R}\right)^{\frac{1}{2}}$.

[3]

MP is a tangent to the earth's surface. $\therefore \angle OPM = 90^\circ$ (radius $\perp$ tangent)

$$x^2 + R^2 = (h+R)^2 \quad (\text{Pythagoras Theorem})$$

$$x^2 = h^2 + 2hR + R^2 - R^2$$

$$x^2 = 2hR \left(\frac{h}{2R} + 1 \right)$$

$$x = (2hR)^{\frac{1}{2}} \left(1 + \frac{h}{2R} \right)^{\frac{1}{2}} \quad (\text{shown})$$

9(b)

It is given that h is small compared to R . If $\alpha = \frac{h}{R}$, show that $\sin \theta \approx (2\alpha)^{\frac{1}{2}} \left(1 - \frac{3}{4}\alpha \right)$.

[5]

$$\sin \theta = \frac{x}{h+R} = \frac{(2hR)^{\frac{1}{2}} \left(1 + \frac{h}{2R} \right)^{\frac{1}{2}}}{R \left(\frac{h}{R} + 1 \right)} = \frac{\frac{1}{R} (2hR)^{\frac{1}{2}} \left(1 + \frac{h}{2R} \right)^{\frac{1}{2}}}{\left(\frac{h}{R} + 1 \right)}$$

$$= \frac{\left(\frac{2hR}{R^2} \right)^{\frac{1}{2}} \left(1 + \frac{h}{2R} \right)^{\frac{1}{2}}}{\left(\frac{h}{R} + 1 \right)} = \frac{\left(\frac{2h}{R} \right)^{\frac{1}{2}} \left(1 + \frac{h}{2R} \right)^{\frac{1}{2}}}{\left(\frac{h}{R} + 1 \right)} \quad \text{correct setup.}$$

$$= (2\alpha)^{\frac{1}{2}} \left(1 + \frac{1}{2}\alpha \right)^{\frac{1}{2}} (\alpha + 1)^{-1} \quad \text{sub } \frac{h}{R} = \alpha$$

$$= (2\alpha)^{\frac{1}{2}} \left[1 + \frac{1}{2} \left(\frac{1}{2}\alpha \right) + \dots \right] \left[1 + (-1)\alpha + \dots \right] \quad \text{apply binomial expansion}$$

$$= (2\alpha)^{\frac{1}{2}} \left(1 + \frac{1}{4}\alpha + \dots \right) (1 - \alpha + \dots)$$

$$= (2\alpha)^{\frac{1}{2}} \left(1 - \alpha + \frac{1}{4}\alpha + \dots \right) \quad \text{correct expansion and simplification}$$

$$= (2\alpha)^{\frac{1}{2}} \left(1 - \frac{3}{4}\alpha + \dots \right)$$

$$\approx (2\alpha)^{\frac{1}{2}} \left(1 - \frac{3}{4}\alpha \right) \quad (\text{since } h \text{ is small compared to } R, \text{ so } \alpha \text{ is sufficiently small. shown.})$$

9(c)

The man M has a scientific instrument which enables him to estimate the angle between PM and the horizontal. Given that this angle is 2° and that the radius of the earth is 6375 km, find estimates for the values of α and h . [4]

$\triangle MOP$ and $\triangle MPQ$ are similar, since $\angle OMP = \angle PMQ$ (common)
and $\angle MPO = \angle MQP = 90^\circ$ (AA similarity Test)

$\therefore \theta = \angle MOP = \angle MPQ = 2^\circ$ ($\angle$ between PM and the horizontal)

$$\Rightarrow \sin 2^\circ = (2\alpha)^{\frac{1}{2}} \left(1 - \frac{3}{4}\alpha\right) \quad (\text{where } \alpha \text{ is sufficiently small})$$

$$(2\alpha)^{\frac{1}{2}} \left(1 - \frac{3}{4}\alpha\right) - \sin 2^\circ = 0$$

Using the G.C.,

$$\alpha = 6.10 \times 10^{-4} \text{ (3 s.f.)}$$

$$\text{or } \alpha = 1.30 \text{ (3 s.f.)}$$

(reject, since α is small)

$\alpha = 1.30$ would make

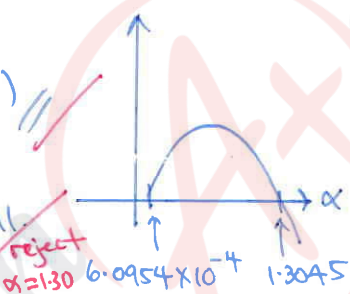
$h = 1.30 R$ which means

that h is not small compared to R)

$$\text{Given } R = 6375 \text{ and } \alpha = 6.0954 \times 10^{-4}$$

$$h = R\alpha = 6375 \times (6.0954 \times 10^{-4})$$

$$= 3.89 \text{ (3 s.f.)}$$



optional

10(a)

The point A has coordinates $(0, 2, -3)$. The line l has equation $\frac{x}{2} = \frac{y+1}{-3} = \frac{z-2}{1}$.

Find the cartesian equation of the plane π which contains A and is perpendicular to l .

[3]

$$\vec{OA} = \begin{pmatrix} 0 \\ 2 \\ -3 \end{pmatrix}, \quad l: \vec{r} = \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}, \lambda \in \mathbb{R}$$

Since l is perpendicular to π , $\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$ is a normal to π . identified normal to π

$$\text{Vector equation of } \pi: \vec{r} \cdot \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = -6 - 3 = -9.$$

$$\therefore \text{Cartesian equation of } \pi: 2x - 3y + z = -9.$$

$$\text{Accept } -2x + 3y - z = 9.$$

10(b)

Hence, or otherwise, find the coordinates of the point P on l which is closest to A .

[3]

For P on l to be closest to A ,

$$AP \perp l \Rightarrow \vec{AP} \cdot \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = 0$$

Since P lies on l ,

$$\therefore \vec{OP} = \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \text{ for some } \lambda$$

$$= \begin{pmatrix} 2\lambda \\ -1-3\lambda \\ 2+\lambda \end{pmatrix}$$

$$\vec{AP} = \vec{OP} - \vec{OA} = \begin{pmatrix} 2\lambda \\ -1-3\lambda \\ 2+\lambda \end{pmatrix} - \begin{pmatrix} 0 \\ 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 2\lambda \\ -3-3\lambda \\ 5+\lambda \end{pmatrix}$$

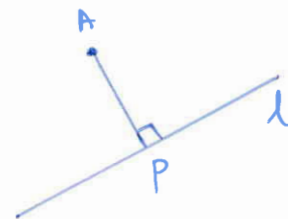
$$\vec{AP} \cdot \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = 0$$

$$\begin{pmatrix} 2\lambda \\ -3-3\lambda \\ 5+\lambda \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = 0$$

$$4\lambda - 3(-3-3\lambda) + (5+\lambda) = 0 \Rightarrow \lambda = -1$$

$$\therefore \vec{OP} = \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}$$

$$\therefore P(-2, 2, 1)$$



Alternatively,

sub eqn of l into π

$$\begin{pmatrix} 2\lambda \\ -1-3\lambda \\ 2+\lambda \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = -9$$

$$4\lambda + 3 + 9\lambda + 2 + \lambda = -9$$

$$\Rightarrow \lambda = -1$$

10(c)

The line m passes through the point with coordinates $(4, -5, 10)$ and P . The line n lies in the same plane as l and m . Find a cartesian equation for n if n is the reflection of the line m the line l . [6]

Let $Q(4, -5, 10)$ and the point of reflection of Q on line l be Q' .

$$\vec{OQ} = \begin{pmatrix} 4 \\ -5 \\ 10 \end{pmatrix}$$

Let F be the foot of the perpendicular of Q on l .

Since F lies on l ,

$$\vec{OF} = \begin{pmatrix} 2\lambda \\ -1-3\lambda \\ 2+\lambda \end{pmatrix} \text{ for some } \lambda$$

$$\vec{QF} = \vec{OF} - \vec{OQ} = \begin{pmatrix} 2\lambda \\ -1-3\lambda \\ 2+\lambda \end{pmatrix} - \begin{pmatrix} 4 \\ -5 \\ 10 \end{pmatrix} = \begin{pmatrix} 2\lambda-4 \\ -4-3\lambda \\ \lambda-8 \end{pmatrix}$$

$$\vec{QF} \perp l \Rightarrow \vec{QF} \cdot \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = 0$$

$$\begin{pmatrix} 2\lambda-4 \\ -4-3\lambda \\ \lambda-8 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = 0$$

$$4\lambda - 8 - 12 + 9\lambda + \lambda - 8 = 0 \Rightarrow \lambda = 2$$

$$\therefore \vec{OF} = \begin{pmatrix} 4 \\ -1-6 \\ 2+2 \end{pmatrix} = \begin{pmatrix} 4 \\ -7 \\ 4 \end{pmatrix}$$

Using the Ratio Theorem,

$$\vec{OF} = \frac{\vec{OQ} + \vec{OQ'}}{2} \Rightarrow \vec{OQ'} = 2\vec{OF} - \vec{OQ} = 2\begin{pmatrix} 4 \\ -7 \\ 4 \end{pmatrix} - \begin{pmatrix} 4 \\ -5 \\ 10 \end{pmatrix} = \begin{pmatrix} 4 \\ -9 \\ -2 \end{pmatrix}$$

$$\vec{Q'P} = \vec{QP} - \vec{OQ'} = \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix} - \begin{pmatrix} 4 \\ -9 \\ -2 \end{pmatrix} = \begin{pmatrix} -6 \\ 11 \\ 3 \end{pmatrix}$$

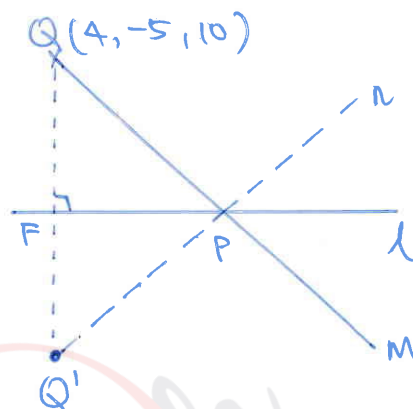
$$\therefore \text{direction vector of } n = \begin{pmatrix} -6 \\ 11 \\ 3 \end{pmatrix}$$

$$\text{Vector equation of } n: \vec{r} = \begin{pmatrix} 4 \\ -9 \\ -2 \end{pmatrix} + \mu \begin{pmatrix} -6 \\ 11 \\ 3 \end{pmatrix}, \mu \in \mathbb{R}.$$

$$\therefore \text{Cartesian equation of } n: \frac{x-4}{-6} = \frac{y+9}{11} = \frac{z+2}{3}$$

$$\text{Accept } \frac{x+2}{-6} = \frac{y-2}{11} = \frac{z-1}{3}$$

or equivalent.



11(a)

The variable A is such that the rate of increase of A with respect to time t is proportional to the product of A and $(10 - A)$. The initial value of A is 2 and when $t = 5$ the value of A is 4.

Write down a differential equation expressing the relation between A and t . Find the time at which $A = 8$, giving your answer correct to 2 decimal places. [8]

Let $\frac{dA}{dt} = KA(10-A)$ where K is a positive constant.
 correct differential //

$$\int \frac{1}{A(10-A)} dA = \int K dt$$

By cover-up rule, $\frac{1}{A(10-A)} = \frac{1}{10A} + \frac{1}{10(10-A)}$

$$\frac{1}{10} \int \frac{1}{A} + \frac{1}{10-A} dA = \int K dt$$

 obtained correct partial fractions

$$\frac{1}{10} (\ln|A| - \ln|10-A|) = Kt + C, \text{ where } C \text{ is an arbitrary const.}$$

 correct integration

$$\ln \left| \frac{A}{10-A} \right| = 10Kt + 10C$$

$$\left| \frac{A}{10-A} \right| = e^{10Kt + 10C}$$

$$\frac{A}{10-A} = \pm e^{10C} \cdot e^{10Kt} = Be^{10Kt} \text{ where } B = \pm e^{10C}.$$

 resolved ln and modulus

When $t=0$ and $A=2$, $\frac{2}{10-2} = Be^0 \Rightarrow B = \frac{2}{8} = \frac{1}{4}$

When $t=5$ and $A=4$, $\frac{4}{10-4} = \frac{1}{4} e^{10K(5)}$

$$\frac{16}{6} = e^{50K}$$

$$\ln \frac{8}{3} = 50K \Rightarrow K = \frac{1}{50} \ln \frac{8}{3}$$

$$\therefore \frac{A}{10-A} = \frac{1}{4} e^{\frac{t}{5} \ln \frac{8}{3}}$$

or $K = 0.019617$ (5sf)

when $A = 8$,

$$\frac{8}{10-8} = \frac{1}{4} e^{\frac{t}{5} \ln \frac{8}{3}}$$

 correct substitution.

$$\ln 16 = \frac{t}{5} \ln \frac{8}{3}$$

$$t = \frac{5 \ln 16}{\ln \frac{8}{3}} = 14.13 \text{ (2 d.p.)}$$

11(b)

Find the value of A when $t = 24$, giving your answer correct to 2 decimal places.

[2]

when $t = 24$,

$$\frac{A}{10-A} = \frac{1}{4} e^{\frac{24}{5} \ln \frac{8}{3}} = 27.7070 \text{ (4 d.p.)}$$

$$A = 277.070 - 27.7070A$$

$$28.7070A = 277.070$$

$$\therefore A = \frac{277.070}{28.7070} = 9.65 \text{ (2 d.p.)}$$

correct simplification (Accept 6.c. solving)

11(c)

Write the solution of the differential equation in the form $A = f(t)$ and sketch the graph of A against t .

[4]

$$\frac{A}{10-A} = \frac{1}{4} e^{\frac{t}{5} \ln \frac{8}{3}}$$

using power law of logarithm.

$$\frac{4A}{10-A} = e^{(\ln \frac{8}{3}) \frac{t}{5}}$$

$$\frac{4A}{10-A} = \left(\frac{8}{3}\right)^{\frac{t}{5}}$$

$$4A = 10\left(\frac{8}{3}\right)^{\frac{t}{5}} - A\left(\frac{8}{3}\right)^{\frac{t}{5}}$$

simplification.

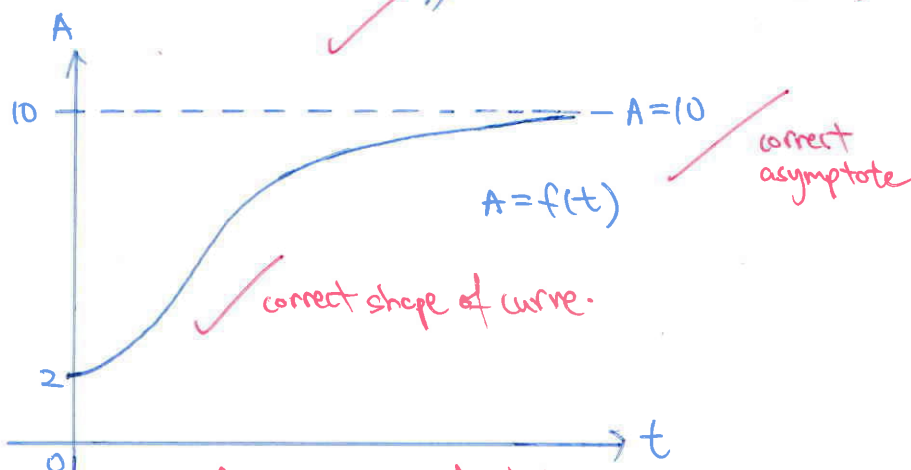
$$\left[4 + \left(\frac{8}{3}\right)^{\frac{t}{5}}\right] A = 10\left(\frac{8}{3}\right)^{\frac{t}{5}}$$

$$A = \frac{10\left(\frac{8}{3}\right)^{\frac{t}{5}}}{4 + \left(\frac{8}{3}\right)^{\frac{t}{5}}}$$

Accept either

$$A = \frac{10}{4e^{-0.196t} + 1}$$

or equivalent.
(A must be the subject)



Note: If curve is drawn for $t < 0$, deduct 1 mark, since t represents time and time cannot be negative.